Project systems theory

22/01/2015, Thursday, 08:30-11:30

You are **NOT** allowed to use any type of calculators.

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1 (10 + 10 = 20 pts) **Linearization**

A simplified model of a patient in the presence of an infectious disease is described by the equations

$$\dot{\xi} = 1 - \xi - \alpha\xi\eta \tag{1a}$$

$$\dot{\eta} = \alpha\xi\eta - \eta - \beta\eta \tag{1b}$$

in which ξ represents the number of non-infected cells, η represents the number of infected cells, α represents the effect of the therapy and $\beta \in (0, 1)$ represents the action of the immune system.

(a) Consider a nonlinear system of the form $\dot{x}(t) = f(x(t))$. A (constant) vector $\bar{x}$ is called an *equilibrium point* if $f(\bar{x}) = 0$. Determine the two equilibrium points of the system (1). Show that one equilibrium corresponds to a healthy patient, i.e. the number of infected cells is zero, and one equilibrium corresponds to an ill patient, i.e. the number of infected cells is non-zero.

(b) Write the linearized models of the system (1) around the two equilibrium points.

2 (20 pts) **Routh-Hurwitz criterion**

Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -a & -a & -2 \end{bmatrix}.$$

Determine all values of a such that the system $\dot{x} = Ax$ stable?

3 (5+10+10=25 pts) **Controllability and feedback**

Consider the system $\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u.$

(a) Is it controllable?

(b) Find a nonsingular T such that

$$T^{-1}AT = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix} \qquad T^{-1}b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for some real numbers α and β.

(c) By using the matrix T of the previous subproblem, find a state feedback of the form $u = k^T x$ such that the closed loop system has poles at -1 and -2.

4 (25 pts) **Output controllability**

Consider the system Σ given by the equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t).$$

with $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$, is called *output-controllable* if for any $x_0 \in \mathbb{R}^n$ and $y_1 \in \mathbb{R}^p$ there exists an input u and a positive real number T such that $y_u(T, x_0) = y_1$. Show that Σ is output-controllable if and only if

$$\operatorname{rank}\begin{pmatrix} D & CB & CAB & \cdots & CA^{n-1}B \end{pmatrix} = p.$$

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10 pts free